

# About the Almeida-Thouless transition line in the Sherrington-Kirkpatrick mean field spin glass model

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In this short note, we consider the Sherrington-Kirkpatrick mean field spin glass model [1, 2] and we prove that, in the thermodynamic limit  $N \rightarrow \infty$ , the quenched free energy per site is strictly greater than the corresponding replica symmetric approximation [1], for values of temperature and magnetic field below the Almeida-Thouless line [3]. This is a simple consequence of rigorous bounds, discovered by F. Guerra [4], which relate the true quenched free energy to the Parisi solution with replica symmetry breaking [5].

Consider the system at temperature  $\beta^{-1}$  and magnetic field  $h$ , and recall that the Almeida-Thouless critical line is defined by the condition

$$\beta^2 \int d\mu(z) \frac{1}{\cosh^4(z\beta\sqrt{\bar{q}} + \beta h)} = 1, \quad (1)$$

where  $d\mu(z)$  is a unit centered Gaussian measure and the Sherrington-Kirkpatrick order parameter  $\bar{q}(\beta, h)$  is the unique [6] solution of

$$\bar{q} = \int d\mu(z) \tanh^2(z\beta\sqrt{\bar{q}} + \beta h). \quad (2)$$

The Parisi solution [5] is defined as

$$\bar{\alpha}_P(\beta, h) = \inf_{x \in \mathcal{X}} \bar{\alpha}(\beta, h; x), \quad (3)$$

where  $\mathcal{X}$  is the space of functional order parameters, i.e., of non decreasing functions

$$x : q \in [0, 1] \rightarrow x(q) \in [0, 1],$$

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and  $\bar{\alpha}(\beta, h; x)$  is defined as

$$\bar{\alpha}(\beta, h; x) = \ln 2 + f(0, h; x, \beta) - \frac{\beta^2}{2} \int_0^1 q x(q) dq. \quad (4)$$

$f(q, y; x, \beta)$  is the solution of the antiparabolic equation

$$\partial_q f(q, y; x, \beta) + \frac{1}{2} (\partial_y^2 f(q, y; x, \beta) + x(q)(\partial_y f(q, y; x, \beta))^2) = 0 \quad (5)$$

with final condition

$$f(1, y; x, \beta) = \ln \cosh(\beta y). \quad (6)$$

The equation for  $f$  can be easily solved if  $x(q)$  is piecewise constant. For instance, if one takes

$$\begin{cases} x(q) = 0 & q \in [0, \bar{q}] \\ x(q) = 1 & q \in (\bar{q}, 1], \end{cases} \quad (7)$$

one finds that  $\bar{\alpha}(\beta, h, x)$  is the so called replica symmetric solution

$$\bar{\alpha}(\beta, h) = \ln 2 + \frac{\beta^2}{4}(1 - \bar{q})^2 + \int d\mu(z) \ln \cosh(z\beta\sqrt{\bar{q}} + \beta h). \quad (8)$$

One expects the quenched free energy per site  $F_N(\beta, h)$  to be related to the Parisi solution by

$$-\lim_{N \rightarrow \infty} \beta F_N(\beta, h) = \bar{\alpha}_P(\beta, h), \quad (9)$$

where  $N$  is the size of the system. While the rigorous proof of this equality has not yet been fully achieved, one can prove [4] that

$$-\beta F_N(\beta, h) \leq \bar{\alpha}_P(\beta, h), \quad (10)$$

for any value of  $N, \beta, h$ .

In the following, we employ the result (10) to prove that the thermodynamic limit of the quenched free energy is strictly greater than its replica symmetric approximation, below the Almeida-Thouless line:

$$-\beta F(\beta, h) \equiv -\beta \lim_{N \rightarrow \infty} F_N(\beta, h) < \bar{\alpha}(\beta, h), \quad (11)$$

for

$$\beta^2 \int d\mu(z) \frac{1}{\cosh^4(z\beta\sqrt{\bar{q}} + \beta h)} > 1. \quad (12)$$

(The limit in (11) exists, thanks to [7]). To this purpose, one simply needs to show that, if (12) holds, there exists a functional order parameter  $\tilde{x}$  such that  $\bar{\alpha}(\beta, h; \tilde{x}) < \bar{\alpha}(\beta, h)$ .

For instance, we choose

$$\begin{cases} \tilde{x}(q) = 0 & q \in [0, \bar{q}] \\ \tilde{x}(q) = m & q \in (\bar{q}, q] \\ \tilde{x}(q) = 1 & q \in (q, 1], \end{cases} \quad (13)$$

where  $0 \leq m \leq 1$  and  $\bar{q} \leq q \leq 1$  and we denote with  $\bar{\alpha}(\beta, h; m, q)$  the corresponding Parisi function  $\bar{\alpha}(\beta, h; \tilde{x})$ .

Of course, since  $\bar{\alpha}(\beta, h; 1, q) = \bar{\alpha}(\beta, h)$ , it is sufficient to prove that

$$\partial_m \bar{\alpha}(\beta, h; m, q)|_{m=1} > 0,$$

for some  $q$ . First of all,  $\bar{\alpha}(\beta, h; m, q)$  is easily found to be

$$\bar{\alpha}(\beta, h; m, q) = \ln 2 + \frac{\beta^2}{2}(1 - q) - \frac{\beta^2}{4}(1 - q^2 + m(q^2 - \bar{q}^2)) + \quad (14)$$

$$+ \frac{1}{m} \int d\mu(z') \ln \int d\mu(z) \cosh^m(\beta h + \beta z \sqrt{q - \bar{q}} + \beta z' \sqrt{\bar{q}}). \quad (15)$$

Next, we compute the derivative with respect to  $m$ , keeping  $q$  fixed, and we find

$$\begin{aligned} \partial_m \bar{\alpha}(\beta, h; m, q)|_{m=1} &\equiv K(\beta, h; q) \equiv \\ &\equiv -\frac{\beta^2}{4}(q^2 - \bar{q}^2) - \int d\mu(z') \ln \int d\mu(z) \cosh(\beta h + \beta z \sqrt{q - \bar{q}} + \beta z' \sqrt{\bar{q}}) \\ &+ \int d\mu(z') \frac{\int d\mu(z) \cosh(\beta h + \beta z \sqrt{q - \bar{q}} + \beta z' \sqrt{\bar{q}}) \ln \cosh(\beta h + \beta z \sqrt{q - \bar{q}} + \beta z' \sqrt{\bar{q}})}{\int d\mu(z) \cosh(\beta h + \beta z \sqrt{q - \bar{q}} + \beta z' \sqrt{\bar{q}})}. \end{aligned} \quad (16)$$

It is clear that, for  $q \downarrow \bar{q}$ , the integration over  $z$  disappears, and

$$K(\beta, h; \bar{q}) = 0.$$

Therefore, in order to check the sign of  $K(\beta, h; \bar{q})$ , we have to expand around  $q = \bar{q}$ . By performing the first two derivatives with respect to  $q$ , one finds

$$\partial_q K(\beta, h; q)|_{q=\bar{q}} = 0$$

and

$$\partial_q^2 K(\beta, h; q)|_{q=\bar{q}} = -\frac{\beta^2}{2} \left( 1 - \beta^2 \int d\mu(z) \frac{1}{\cosh^4(z\beta\sqrt{\bar{q}} + \beta h)} \right).$$

This computation requires a simple integration by parts on a Gaussian variable. It is clear that, when condition (12) holds,  $\partial_q^2 K(\beta, h; q)|_{q=\bar{q}} > 0$ , so that

$$\partial_m \bar{\alpha}(\beta, h; m, q)|_{m=1} > 0,$$

at least for  $q$  small.

This, together with Guerra's bound (10), completes the proof of the result (11), i.e., of the instability of the replica symmetric solution.

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